USEFUL NUCLEAR DERIVATIONS A

A.1 The semi-empirical mass formula

The mass of a nucleus defined by A and Z is given by

$$M(A, Z) = Z\, m_H + (A - Z)\, m_n - B(A, Z)/c^2, \tag{A.1}$$

where $B(A, Z)$ is the binding energy of the nucleus.

The semi-empirical mass formula, based on the liquid drop model, considers five contributions to the binding energy:

1. The volume term $a_V\, A$. Since the nuclear force is saturated, each nucleon contributes about 16 MeV to the binding of the nucleus.

2. The surface term, which gives the reduction in binding resulting from the reduced binding at the nuclear surface, $-a_S\, A^{2/3}$.

3. The Coulomb term, which represents the Coulomb repulsion of the $Z\,(Z - 1)/2$ pairs of protons in the nucleus. For a spherical nucleus of radius $R = r_0 A^{1/3}$ with the charge spread evenly throughout the sphere the Coulomb energy is

   $$-\frac{3}{5}\frac{1}{4\pi\epsilon_0}\frac{Z(Z-1)e^2}{r_0 A^{1/3}}.$$

   For a general charge distribution not too different from the above, this can be parameterized as $-a_C\, Z^2\, A^{-1/3}$.

4. The asymmetry term, which accounts for the difference between proton and neutron number. If there were no Coulomb interaction between protons, one would expect, from symmetry arguments applied to a Fermi gas, to find equal numbers of protons and neutrons. In order to generate the observed neutron excess (in most nuclei) we need to shift nucleons from the "proton side" to the "neutron side" of these two Fermi gases. These neutrons can only be added above the Fermi level, so energy must be put into the system. This is the asymmetry energy which reduces the nuclear binding.
   The system is symmetrical about $N = Z$; the same energy would be required to shift nucleons the other way if we require a proton excess. Thus to lowest order, one can expect the energy to vary as $(N - Z)^2$; in addition, the Fermi gas energy level spacing varies as $1/A$ so that the asymmetry term is

   $$-a_A \frac{(A - 2\,Z)^2}{A}.$$

5. An empirical term to take into account the observed pairing of nuclei:

$$\delta(A,Z) = \begin{cases} +\delta_0 & \text{Z and N even (A odd)} \\ 0 & \text{A odd} \\ -\delta_0 & \text{Z and N odd (A odd)} \end{cases} .$$

In 1993 there were 342 stable nuclei in the mass compilation, 209 with even A, even Z; 70 with odd A, even Z; 59 with even A, odd Z and only 4 with odd A and Z (^{2}H, ^{6}Li, ^{10}B, ^{14}N). Clearly pairing enhances stability (or binding energy). This can also be seen, for instance, in the neutron separation energies of neighboring isotopes, etc.

The binding energy is thus

$$B(A,Z) = a_V\, A - a_S\, A^{2/3} - a_A \frac{(A-2\,Z)^2}{A} - a_C\, Z^2\, A^{-1/3} + \delta_0(A,Z). \tag{A.2}$$

The coefficients are determined by fitting to a suitably large data set of masses (hence semi-empirical). A typical set is (all values in MeV):

$$a_V = 15.8, \qquad a_V = 18.3, \qquad a_V = 0.714, \qquad a_V = 23.2, \qquad \delta_0 = \frac{12}{A^{1/2}}. \tag{A.3}$$

A.2 The line of stability

Greater binding energy per nucleon implies greater stability. It is most convenient to explore this in the context of a set of isobars.

The masses of the members of a set of isobars can be obtained by rearranging the semiempirical mass formula A.1:

$$M(A,Z) = \alpha\, A - \delta(A,Z) + \beta\, Z + \gamma\, Z^2,$$

where

$$\begin{aligned} \alpha &= m_n\, c^2 - a_V + a_A + a_S^{-1/3} \\ \beta &= (m_H - m_n)\, c^2 - 4a_A - a_C\, A^{-1/3} \\ \gamma &= a_C\, A^{1/3} + 4a_A/A \end{aligned} .$$

This equation has the form of a parabola for fixed A; we can solve for the value of Z giving the greatest binding energy (smallest mass), i.e. the most stable isobar. Thus

$$\frac{\partial}{\partial Z} M(A,Z) = 0$$

yields

$$Z_S = \frac{-\beta}{2\gamma} = \frac{A/2 + (m_n - m_H)c^2 A/8a_A + a_C A^{2/3}/8a_A}{1 + \frac{1}{4}(a_C/a_A)A^{2/3}}. \tag{A.4}$$

Inserting the values for the coefficients A.3 and rearranging,

$$Z_S = \frac{A\left(1 + 0.0077\, A^{-1/3}\right)}{2 + 0.0154\, A^{2/3}}. \tag{A.5}$$

This then gives the equation for the "valley of stability" on the (N,Z) chart of nuclides. Note that is determined by an interplay between the Coulomb force (makes Z a minimum) and the asymmetry term (makes $N = Z$).

In Fig. A.1 the curve coming from Eq. A.6 is superimposed on the experimental valley of stability. The agreement for the most frequent isotope is very good, with an error of maximum 1 proton (1 neutron).

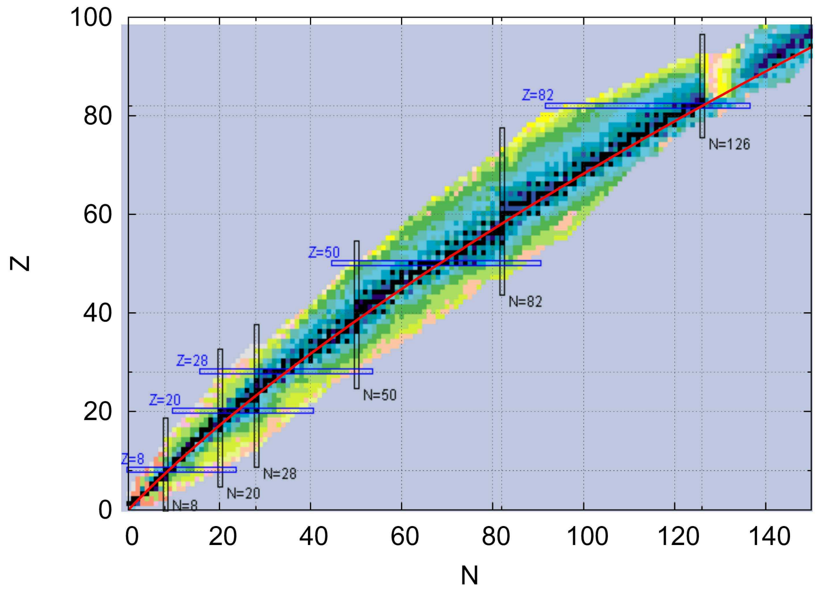

Figure A.1: The valley of stability. The red curve comes from Eq. A.6 and the background image from http://www.nndc.bnl.gov/nudat2/.

A.3 Coulomb barrier

The Coulomb barrier is the energy barrier resulting from electrostatic interaction that two nuclei must overcome in order that they can approach closely enough to undergo nuclear fusion. The Coulomb barrier is produced by electrostatic potential energy. In the fusion of light elements to form heavier ones the positively charged nuclei must be forced close enough together to cause them to fuse into a single heavier nucleus. The force between nuclei is repulsive until a very small distance separates them, and then it rapidly becomes very attractive. Therefore, in order to surmount the Coulomb barrier and bring the nuclei close together where the strong attractive forces operate, the kinetic energy of the particles must be as high as the top of the Coulomb barrier.

The Coulomb electrostatic potential for two colliding nuclei at radius d could be expressed as

$$U(Z_1, Z_2) = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{d}, \tag{A.6}$$

where $\epsilon_0 = 8.85\times10^{-12}$ C^{2}N^{-1}m^{-2} and $e = 1.60\times10^{-19}$ C.

The nuclear radius is conventionally set to $r(A) = r_0 A^{1/3}$ with $r_0 = 1.44\times10^{-15}$ m. In principle the Coulomb barrier could be overcome if the distance is lower than the sum of the nuclear radii of the two nuclei. Therefore $d = r_0(A_1^{1/3} + A_2^{1/3})$.

Substituting the previous expression in Eq. A.6 it results

$$U(Z_1, Z_2)[\text{MeV}] \simeq \frac{Z_1 Z_2}{A_1^{1/3} + A_2^{1/3}}, \tag{A.7}$$

which is the minimum energy to overcome the barrier in the center-of-mass frame. To obtain the lab frame expression on must multiply by the factor $(A_1 + A_2)/A_2$ resulting

$$U(Z_1, Z_2)[\text{MeV/A}] \simeq \frac{A_1 + A_2}{A_1 A_2}\frac{Z_1 Z_2}{A_1^{1/3} + A_2^{1/3}}. \tag{A.8}$$

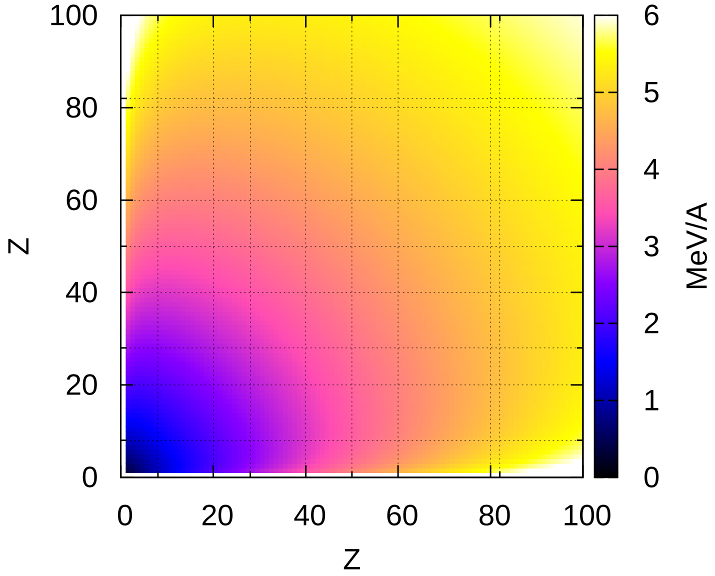

Figure A.2: Coulomb barrier for the most frequent isotopes appearing in the valley of stability.

Combining the previous expression with Eq. A.6 one can obtain the Coulomb barrier for the most frequent isotopes appearing in the valley of stability, plotted in Fig. A.2.